The Simple Pendulum

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10:00 A.M - 1:00 P.M

1 Introduction:

A pendulum is a deceptively simple looking device consisting of a weight (referred to as a bob) tied to a string which is then wrapped around some form of axle, which it is able to freely rotate around. The measurements of these free rotations are referred to as oscillations; which are defined as the time taken for the pendulum bob to rotate in a 2 dimensional plane from a single point and return to that same point. Despite the simple apparatus there are many forces at play on the pendulum including Tension, Gravitation, Restoring Force and Angular Momentum.

It is possible to derive the equation for a simple harmonic oscillator by using Newton's Second Law of Motion:

$$F = ma \tag{1}$$

It is known that the only force present is the Restoring force F_{R}
The Restoring force equation is:

$$F_{\mathrm{R}} = -mgsin\theta \tag{2}$$

because of the nature of this experiment it is known that the initial angle θ is small, and thus we can say $Sin\theta \approx \theta$, the angle θ can be switched out for the two components of the arc length where $x = s = l\theta$ Once these substitutions are made it produces the equation:

$$F_{\mathrm{R}} = -\frac{mg}{l}x \tag{3}$$

From Eq. (3) it is determined that if the bob was released with zero initial velocity through the initial angle or displacement in this case, then there are two solutions that satisfy this:

$$x = x_0\, cos(\omega t), \theta = \theta_0\, cos(\omega t) \tag{4}$$

In these solutions, ω is used to represent the angular frequency of the bob: $\omega = \sqrt{\frac{g}{l}}$, It can also be written ω in terms of the frequency $\omega = 2\pi f$, it is known the frequency is the same as number of oscillations so therefore; $\omega = 2\pi T$, where T = period. Which results in:

$$T = 2\pi\sqrt{\frac{g}{l}}$$

$$\text{where; } x = x_0\, cos(\tfrac{2\pi t}{T}), \theta = \theta_0\, cos(\tfrac{2\pi t}{T}) \tag{5}$$

Square and rearrange Eq. (5) to get:

$$T^2 = \frac{4\pi^2}{g}l \tag{6}$$

As can be seen from Eq. (6) the final resultant equation doesn't have a variable m (mass), therefore it is verified mathematically that the mass of the bob has no affect on the period of the pendulum in the case of this experiment with such a small angle θ.

2 Experimental Details:

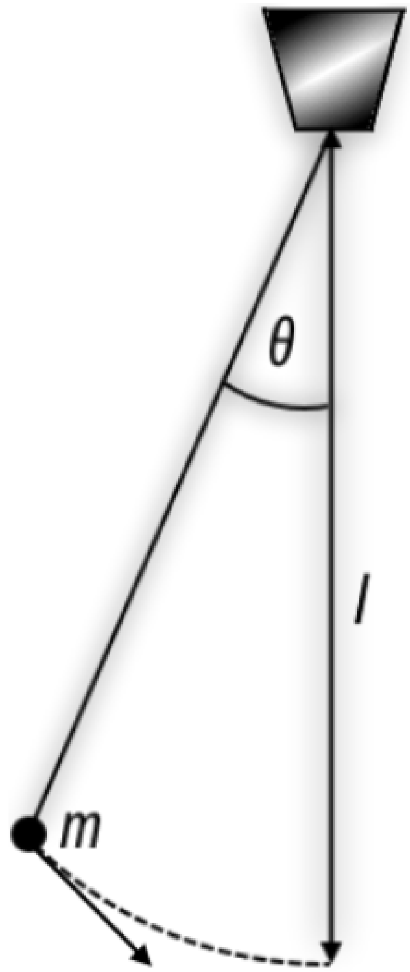

Figure 2.1: Apparatus of Stop Watch .

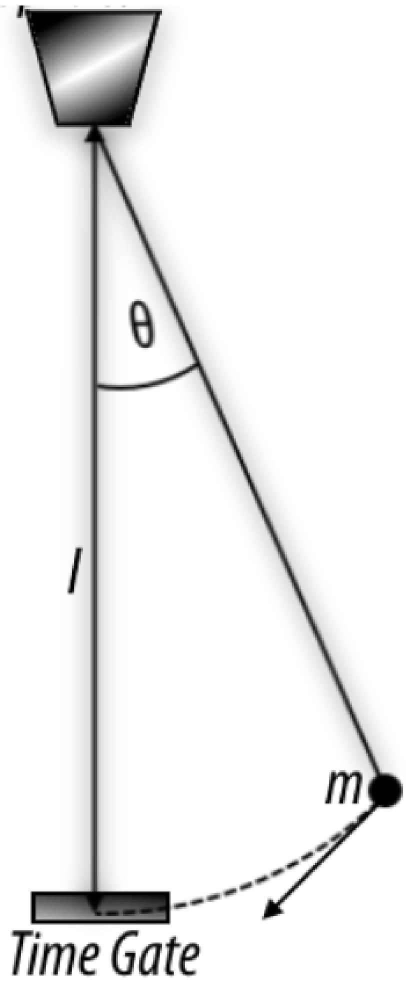

Figure 2.2: Apparatus of Time Gate

The set up of this apparatus consists of a Retort Stand, attached to the Retort Stand is a Split Cork which acts as the axle in this circumstance with a String running through it. At the end of the string is the Weight which is known as the Bob. The string length is measured and changed using a Metre stick and the oscillation time is measured using a stop watch. This is all the equipment used in the first part of the experiment, however for the second part of the experiment a light gate was introduced to measure the oscillation time instead.

3 Results and Data Analysis

Table 1: Average Stopwatch times at each interval (Acceleration)

Length(m)	t(s) for N=20	T(s) (stop watch)	$T^2(s^2)$	T (s) (Time Gate)	$T^2$2 (s^2)
0.20	20.31	1.02	1.04	0.89	0.79
0.30	21.28	1.06	1.12	1.12	1.25
0.40	25.30	1.27	1.61	1.26	1.59
0.50	27.96	1.39	1.93	1.42	2.02
0.60	30.14	1.51	2.28	1.54	2.37
0.70	33.89	1.69	2.86	1.66	2.76
0.80	36.32	1.81	3.28	1.70	2.89
0.90	38.36	1.92	3.69	1.93	3.72
1	40.30	2.00	4.00	2.02	4.08
1.10	41.83	2.08	4.33	2.04	4.16
1.20	43.30	2.17	4.71	2.18	4.75
1.30	45.34	2.27	5.15	2.29	5.24

Using the equations:

$$\frac{Number\ of\ Oscillations(N)}{Time\ Taken\ To\ Complete(s)} = f \quad \text{and} \quad \frac{1}{f} = T \qquad (7)\ ,\ (8)$$

$$\frac{20}{20.31} = 0.98\ ,\ \frac{1}{0.98} = 1.02$$

$$\frac{20}{21.28} = 0.94\ ,\ \frac{1}{0.94} = 1.06$$

$$\frac{20}{25.30} = 0.79\ ,\ \frac{1}{0.79} = 1.27$$

$$\frac{20}{27.96} = 0.72\ ,\ \frac{1}{0.72} = 1.39$$

$$\frac{20}{30.14} = 0.66\ ,\ \frac{1}{0.66} = 1.51$$

$$\frac{20}{33.89} = 0.59\ ,\ \frac{1}{0.59} = 1.69$$

$$\frac{20}{36.32} = 0.55\ ,\ \frac{1}{0.55} = 1.81$$

$$\frac{20}{38.36} = 0.52\ ,\ \frac{1}{0.52} = 1.92$$

$$\frac{20}{40.30} = 0.50\ ,\ \frac{1}{0.50} = 2.00$$

$$\frac{20}{41.83} = 0.48\ ,\ \frac{1}{0.48} = 2.08$$

$$\frac{20}{43.30} = 0.46\ ,\ \frac{1}{0.46} = 2.17$$

$$\frac{20}{45.34} = 0.44\ ,\ \frac{1}{0.44} = 2.27$$

Graphing the values for T^2 vs Length for both the Stop Watch values and Time gate values:

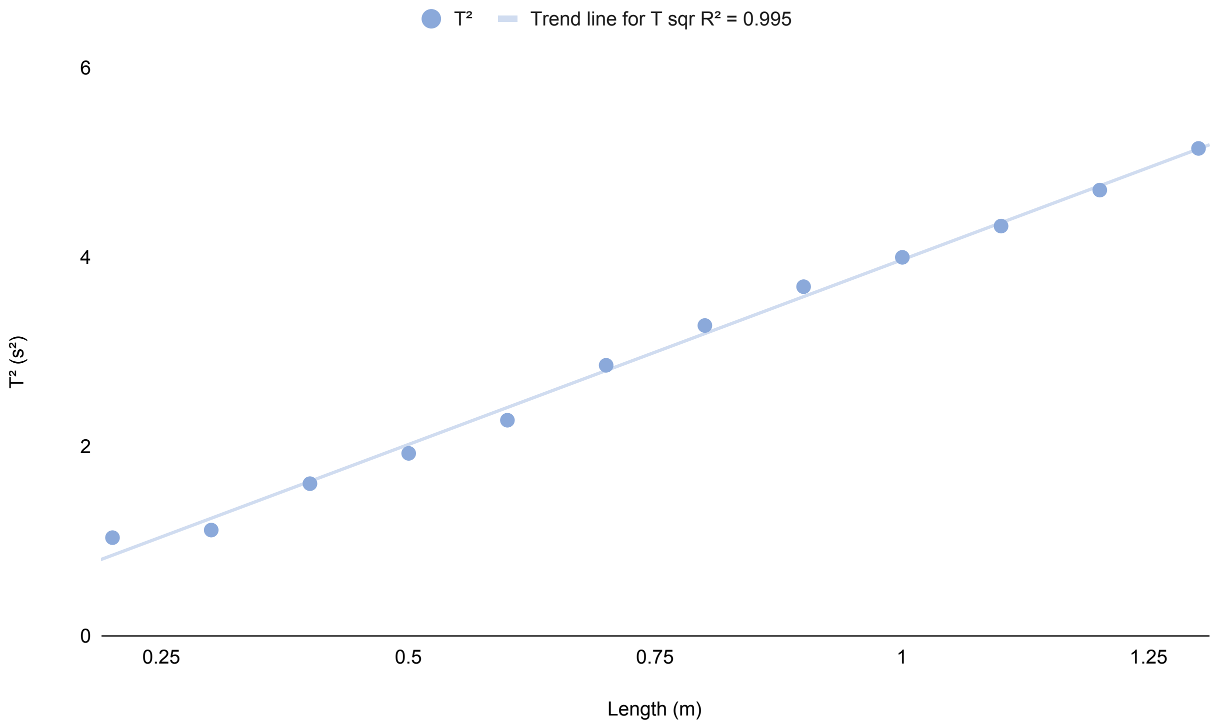

Figure 3.1: T^2 vs Length (Stop Watch)

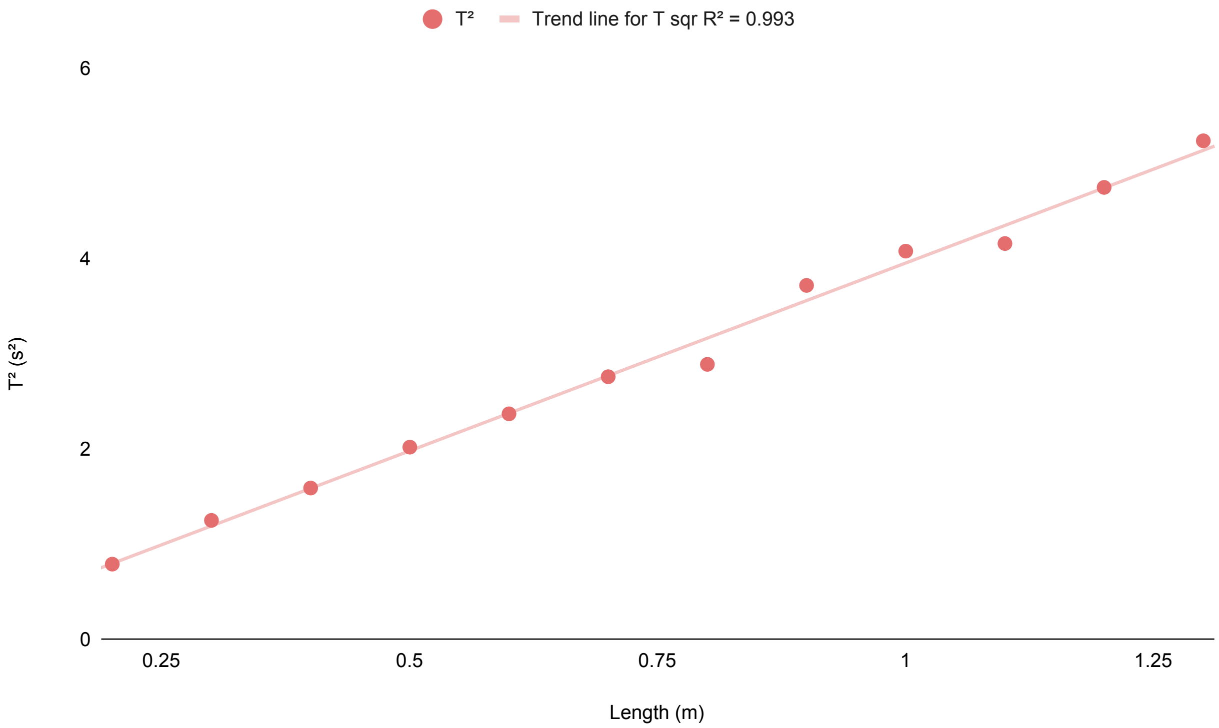

Figure 3.2: T^2 vs Length (Time Gate)

Now comparing the equations of the lines to Eq. (6)

$$y = mx + c \tag{9}$$

Where y = T^2, $m = \frac{4\pi^2}{g}$, x = Length and c = 0 Therefore to calculate g using the slope, use $m = \frac{4\pi^2}{g}$ which can be manipulated to get:

$$g = \frac{4\pi^2}{m} \tag{10}$$

In order to be able to calculate a value for g, a value for the slope is needed. Which can be calculated using the slope formula:

$$\frac{y_2 - y_1}{x_2 - x_1} \tag{11}$$

$$\frac{5.15-1.04}{1.30-0.20} = 3.74\ ms^{-2}$$

$$\frac{5.24-0.79}{1.30-0.20} = 4.05\ ms^{-2}$$

Now that there is a value for each of the slopes, they can be subbed in for m in Eq.10 and get 2 values for g:

$$10.56\ ms^{-2} = \frac{4\pi^2}{3.74}$$

$$9.75\ ms^{-2} = \frac{4\pi^2}{4.05}$$

The accuracy of these values can now be compared to the known value of g (9.81 ms^{-2}) Using the following formula:

$$\frac{|New\,Value - Known\,Value|}{Known\,Value}\ x\ 100 \tag{12}$$

$$\frac{|10.56 - 9.81|}{9.81}\ x\ 100 = 7.65\%$$

$$\frac{|9.75 - 9.81|}{9.81}\ x\ 100 = 0.61\%$$

These percentages verify that the values calculated for g are extremely accurate and that as to be expected the measurements using the more precise time gate apparatus gave a much more accurate value for g.

Although the R^2 value is extremely close to 1, and many of the points sit flush on the line of best fit, an acknowledgement must be made that the line of best fit itself has an error associated with it and that error can be mathematically worked out, Fig. 3.3 and 3.4 represent the graphs of Length versus $Period^2$ with a parallelogram expressing the points at each of the four corners while the two lines both represent minimum slopes (m_{min}) and maximum slopes (m_{max})

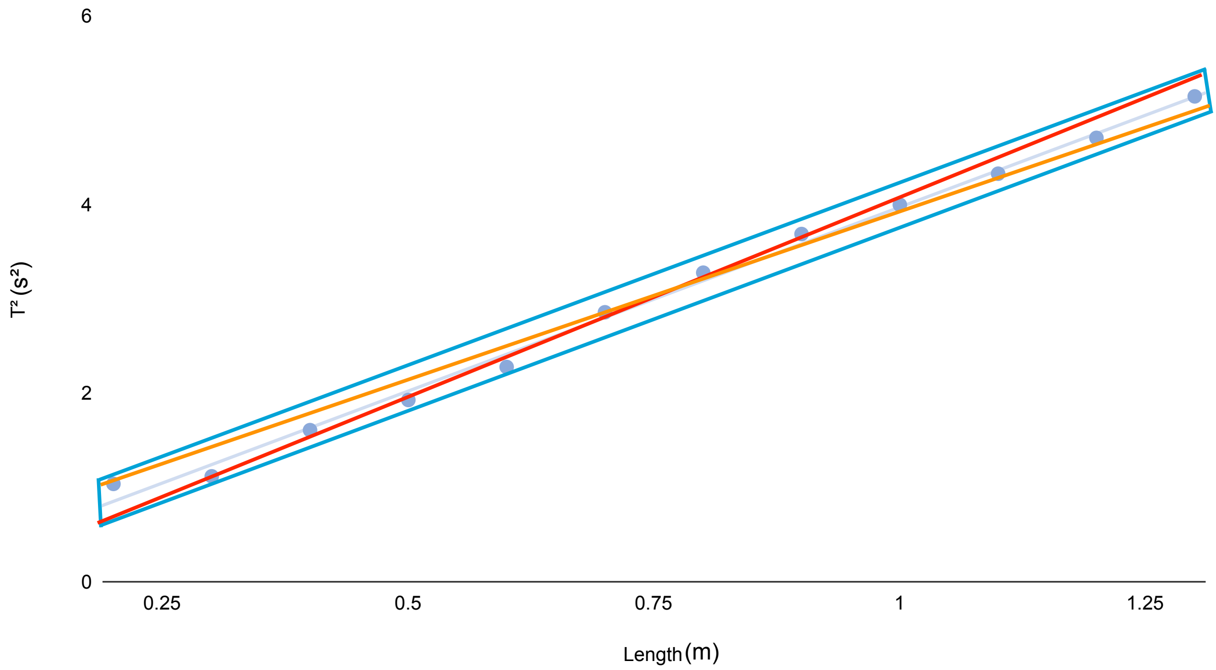

Figure 3.3: T^2 vs Length (Stop Watch) Minimum Maximum slopes

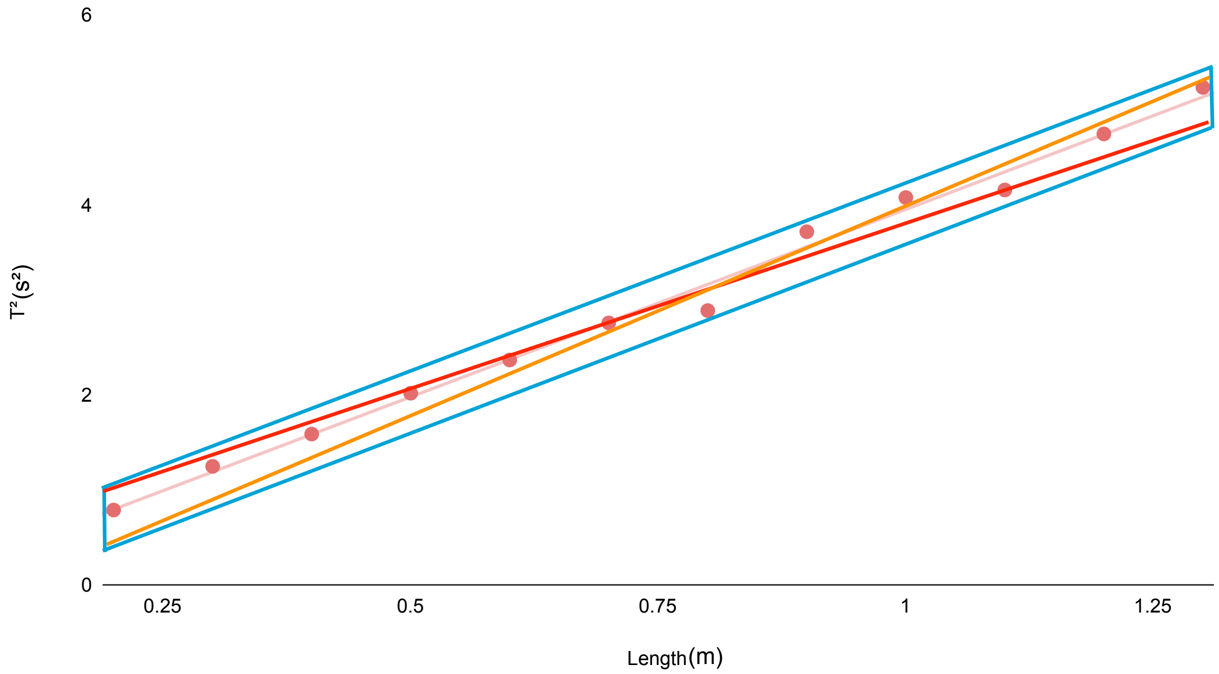

Figure 3.4: T^2 vs Length (Time Gate) Minimum Maximum slopes

and it is possible to mathematically work out what that error is, first determine the slope of the two extreme corners of the parallelogram on the graph, once again using Eq. 11:

$$m_{\max} = \frac{5-0.4}{1.27-0.1} = 3.93\ ms^{-2}$$

$$m_{\min} = \frac{4.5-1.2}{1.28-0} = 3.77\ ms^{-2}$$

The error in the slope can be calculated using

$$\Delta m = \frac{m_{\max} - m_{\min}}{2} \tag{13}$$

$$0.08\ ms^{-2} = \frac{3.93 - 3.77}{2}$$

then get the error relative to the slope using the equation:

$$\frac{\Delta m}{m}\ x\ 100\% \tag{14}$$

$$\frac{0.08}{3.74}\ x\ 100\% = 2.14\%$$

then repeat this process in order to work out a similar value for the second graph:
Using Eq. 11:

$$m_{\max} = \frac{4.1-0.5}{1.27-0.17} = 3.27\ ms^{-2}$$

$$m_{\min} = \frac{4-0.75}{1.26-0.19} = 3.04\ ms^{-2}$$

Using Eq. 13:

$$0.115\ ms^{-2} = \frac{3.27 - 3.04}{2}$$

And finally using Eq. 14 to get a percentage relative error:

$$\frac{0.115}{4.05}\ x\ 100\% = 2.84\%$$

$$Values : 2.14\%\ ,\ 2.84\%$$

From these two values it is concluded that the relative error of the lines of best fits are low, however; despite using more precise measuring apparatus and giving a more accurate value for g, the line of best fit for the time gate measurements has a slightly higher error associated with it. This can be seen visually as Fig. 3.4 has a larger parallelogram surrounding the points compared to Fig 3.3.

Up until now all of the results expressed in this report have been achieved using small angles ($\theta < 15°$), however it is important to also represent what occurs once this parameter is changed. The experiment was repeated following a constant length (m), however this time the angle θ was allowed to exceed 15°. The results were as follows:

Table 2: Period of simple pendulum when $\theta > 15°$

Variable Angle $\theta(°)$	Period
10	1.370
15	1.372
20	1.373
25	1.375
30	1.386
35	1.387
40	1.388
45	1.388
50	1.394
55	1.396
60	1.402
65	1.404
70	1.421
75	1.421

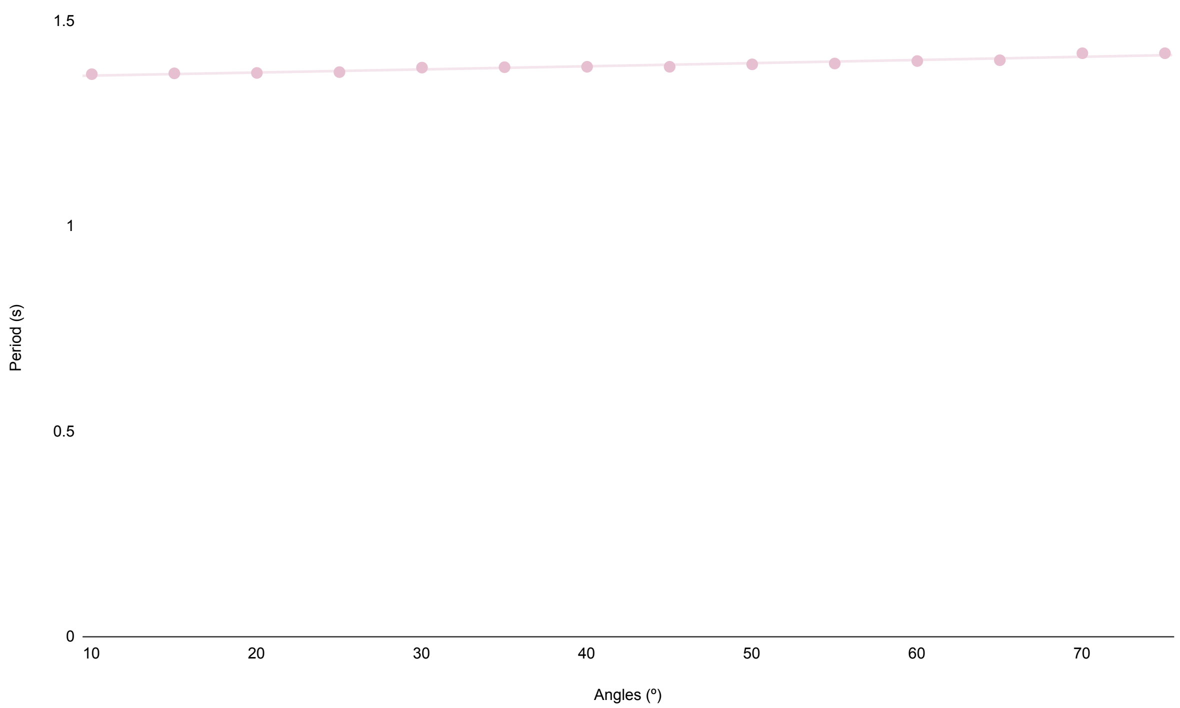

Figure 3.5: Period vs Angle

Fig. 3.5 represents a straight line with a slight increase as the angle increases.

Despite Fig. 3.5 representing an almost complete straight line with a very small slope it is quite well documented in the field of physics that such increases in angles represents an exponentially increasing period.[1][2] It is also well documented that an entirely new equation is required for pendulum periods using large angles.[3]

Although this report wont be going into the details of the new equations in the citation, an explanation can be given as to why it is needed. During the introduction of this report while deriving the equation of Harmonic Motion using Newton's Second Law, the claim was made that said '$Sin\theta \approx \theta$' this approximation held true due to the fact that there was small angles being used during the experiment, however once the angle is increased beyond 15° - this approximation is no longer maintained.

If this experiment was to be repeated with the sole intent of calculating an accurate value of g, it would be recommended that the Stop Watch method be discarded in favour of the Time Gate set-up which produced much more accurate results for g.

An improvement which could be made on this lab report would be to repeat the variable angle portion of the experiment and obtain measurements for even more angles in order to hopefully get the exponential graph which we expect to see based on our mathematical principals and prior documented accounts.

4 Conclusion:

An equation for harmonic oscillation was successfully derived using Newton's Second Law of Motion, two values for g were determined 10.56 ms^{-2} and 9.75 ms^{-2} with calculated inaccuracies of 7.65% and 0.61% respectively. Line of best fit errors were calculated to be 2.14% and 2.84%. It was determined that The Simple Pendulum only gives accurate measurements for our derived harmonic oscillation equation when $\theta < 15°$.

5 References:

[1] "Simulating a Pendulum - ScienceBlogs.", May 16, 2013
https://scienceblogs.com/principles/2013/05/16/simulating-a-pendulum.

[2] "The Large-Angle Pendulum Period - AAPT.", 14 February 2003
http://dx.doi.org/10.1119/1.1557505.

[3] "A simple formula for the large-angle pendulum period - AAPT.", 15 April 2002
http://dx.doi.org/10.1119/1.1457310.