

Computational Lab Report

Graphical Methods

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Abstract

Python is extremely powerful programming language for physicists and mathematicians, not just for its ability to carry out complex maths equations but also to be able to graph representations for these mathematical equation. We are going to be using quiver plots to graphically represent the potential solutions to differential equations. In this report we will be including plots containing vector fields with directions and magnitudes as well as contour plots which gives a background color representing the magnitude. These plots represent real physical systems specifically population, Lotka-Volterra model and limit cycles. We do this to explore the different graphical methods of displaying differential equations and when they don't accurately represent the real world models they are trying to reproduce.

1 Introduction

This report will be building on what we have previously done regarding Euler's method and more importantly the Runge-Kutta fourth order method of solving differential equations in python, however we will be moving away from these methods and focusing on quiver plots which are used to produce vector fields. These quiver plots will graphically describe the solution to differential equations at many different points this is extremely useful for representing different types of differential equations. In this lab report we will graphically be describing population simulations from differential equations. The two population models are known as the Verhulst model and the Lotka-Volterra model respectively. In this report we will be commenting on the difference between two graphical models as well as how these graphical differences represent the theoretical differences inherent in the two models. We will then discuss some more mathematically abstract models which we believe will help us learn more about graphical methods in Python.

2 Background and Theory

The first thing we cover in this report is the theory surrounding the quiver plots produced in Python. Vector fields describe the direction and magnitude of vectors at a number of points in a plane, vector fields are common in electromagnetism and also in a field of mathematics known as vector calculus. Its no surprise then that Pythons plotting package known as matplotlib comes with the ability to create graphical representations of vector fields included in the package.

Throughout the report we will be making reference to vectors, vectors are any object which has both a direction as well as a magnitude. We often display vectors graphically as arrows where the head of the arrow denotes the direction and the length of the arrow represents the magnitude.

Taking the vector field example one step further we can create a certain type of plot known as a stream plot, this stream plot creates trajectories or field lines from vector fields. Each of the stream lines represents a possible solution to a differential equation, this way we can visualise solutions to differential equations without actually solving them.

In this experiment we will be using quiver plots and stream plots to show the expected behaviour of multiple differential equations, this is because quiver plots and by extension stream plots are python's way of graphing vector fields, quiver plots are extremely useful because we can change some of the initial conditions to change how the vector field behaves, for example one of these conditions is the number of vectors that appear per grid point this allows us to increase or decrease the density of vectors.

The first part of this experiment concerns itself with getting familiar with these initial conditions before returning to the harmonic oscillator and damped harmonic oscillator equations we have worked with previously.

The second part of this experiment concerns itself with graphing vector fields showing the potential solutions for population differential equations. One such equation is the logistic equation known as the Verhulst model for population. The equation is as follows:

$$\frac{dx}{dt} \equiv \dot{x} = rx\left(1 - \frac{x}{K}\right) \quad (1)$$

Where K is the carrying capacity or the maximum population of the system, r is the rate of growth or decay of the system and $x(t)$ represents the population at a certain point in time.

This equation is considered to be a simple model for how population reaches equilibrium and only focuses on one population and doesn't account for how one population may have an impact on another population such as a hunter / prey model which we will explore in more detail.

We will also explore the Lotka-Volterra (LV) Model which models two populations simultaneously and how the two populations impact each other using rabbits and foxes as the example. The equations for the Lotka-Volterra (LV) Model is as follows:

$$\frac{dx}{dt} = ax - bxy \quad (2)$$

$$\frac{dy}{dt} = cxy - dy \quad (3)$$

Where $x(t)$ is the population growth of the rabbits, $y(t)$ is the population growth of the foxes, a defines the growth rate, b is a parameter that measure how effective the foxes are at hunting rabbits, c is a parameter that defines how beneficial rabbits are to the foxes and d is a measure of this fox death rate.

We can see that this model uses two differential equations to represent each population and the population of one species has an impact on the population of the other, more specifically a higher population of foxes reduces the population of the rabbits and the decrease in rabbit populations results in a decreased fox population. When we analyse the graphs obtained from this model we will discuss the model in more detail and its strengths and weaknesses as a biological model to represent population.

This is all the theory we need to begin analysing the data collected from the python programmes in the next section of the report. We will not go into the results collected during the experiment.

3 Results

In the first couple of programmes we are familiarising ourselves with the quiver function in python, we go into why they are important and how they are used as well as going into variations on quiver plots such as streamline plots and situations where they are the better option. We also rethread over previously covered topics such as harmonic motion and displaying harmonic motion graphically using vectors. Starting at Program 4 we get into the population models we discussed in the theory section.

Program 1:

Program 1 revolves around using quiver plots to create vector fields for a specific equation given the initial parameters: range 0 to 2π , 101 grid points the equations are as follows:

$$V(x, y) = (V_x, V_y)$$

$$V_x = \cos(x) * y$$

$$V_y = \sin(x) * x$$

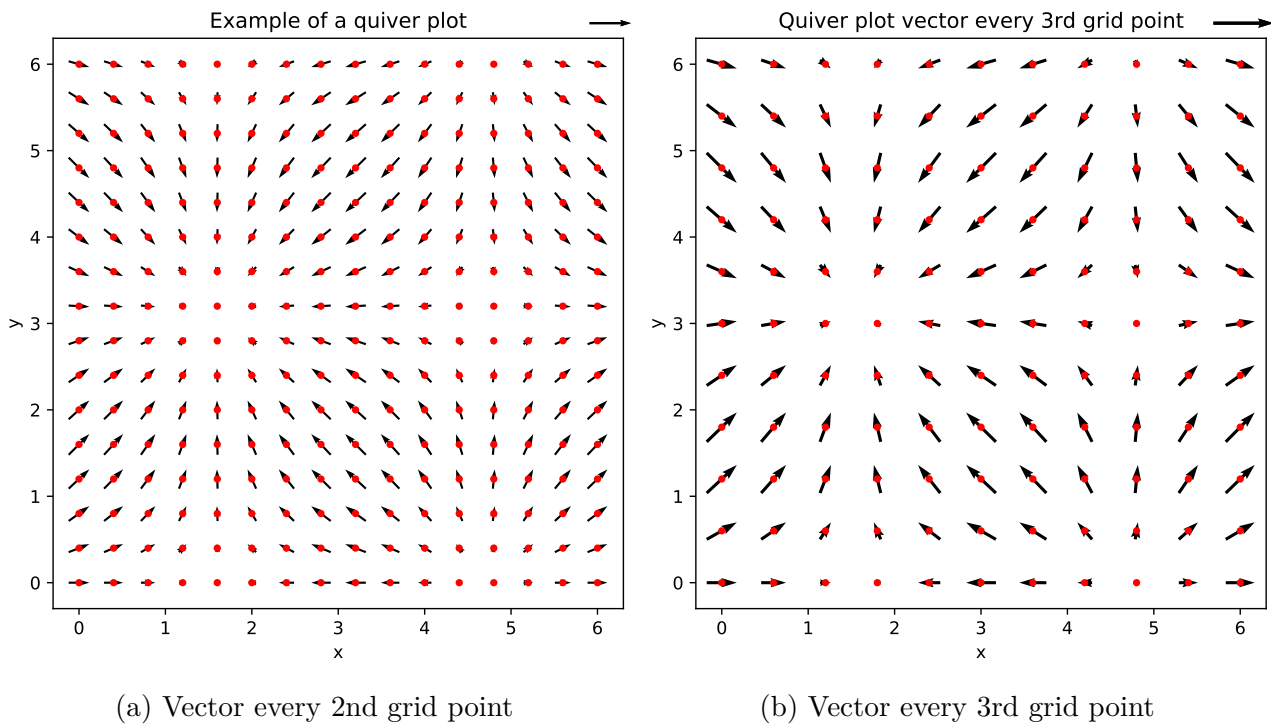


Figure 1: Example of a simple quiver plot

The vector plots for these equations are as follows: The above examples show what a basic quiver plot should look like, during our experiment the only parameters we changed is how often the vectors appear along the grid lines. You can clearly see that by changing the program so that the vectors appear every 3rd grid line reduces the number of vectors present on the graph and can potentially make it easier to read depending on what exactly you are trying to analyse about the plot. Regardless of the number of vectors present both graphs effectively show the potential solutions to the functions they are trying to graphically show, we can see the different directions being expressed by where the heads of the arrows are pointing and the arrows have different lengths with the longest arrows having the biggest magnitude. At both $y = 0$ and $y = 3$ the same pattern seems to appear where the points closest to $(2,0)$, $(2,3)$, $(5,0)$ and $(5,3)$ appear to have no direction or magnitude vectors present and just display the scatter points.

Program 2:

The second program involves modifying the previous code to be a scalar function. We then plotted a contour map gradient on top of the plot of the scalar function, the graphs were as follows:

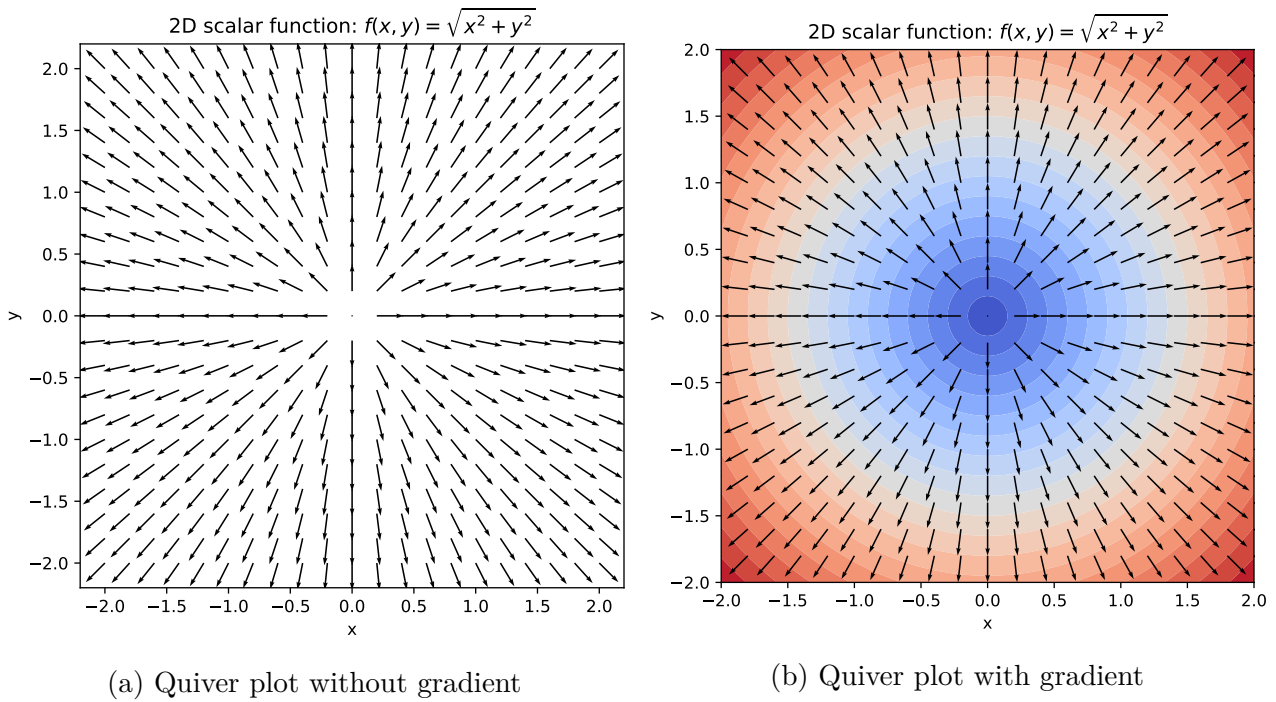


Figure 2: Quiver plot of scalar function $\sqrt{x^2 + y^2}$ (Vector every 5th grid point)

We can see from the graph that even setting the vectors to display every 5th grid point the plot is extremely crowded and if we were working with a function that wasn't so uniform it may actually be difficult to interpret the vector field correctly, this is why the gradient is so helpful it shows a visual representation of where the vectors are directed and the magnitude of those vectors using a color gradient where blue is the lowest concentration and red is the greatest. In the gradient version of the plot it is even easier to see that the plot is split up into four equal sections where the vectors are all pointing towards the four corners of the plot away from some central point at $(0,0)$.

Program 3:

Both of the programs above were just examples of how quiver plots work and the different techniques you can use to try and make the plots easier to read and notice patterns however in this program we will finally be using a real world equation as apposed to a theoretical mathematical vector field. This example is the harmonic oscillator which we have worked with previously.

This is a similar process to the two above examples however this time we will be switching our the equation to fit the equation for a harmonic oscillator and a damped harmonic oscillator respectively. The values chosen were $\omega = 1$ and $b = 0.45$ and the plots were as follows:

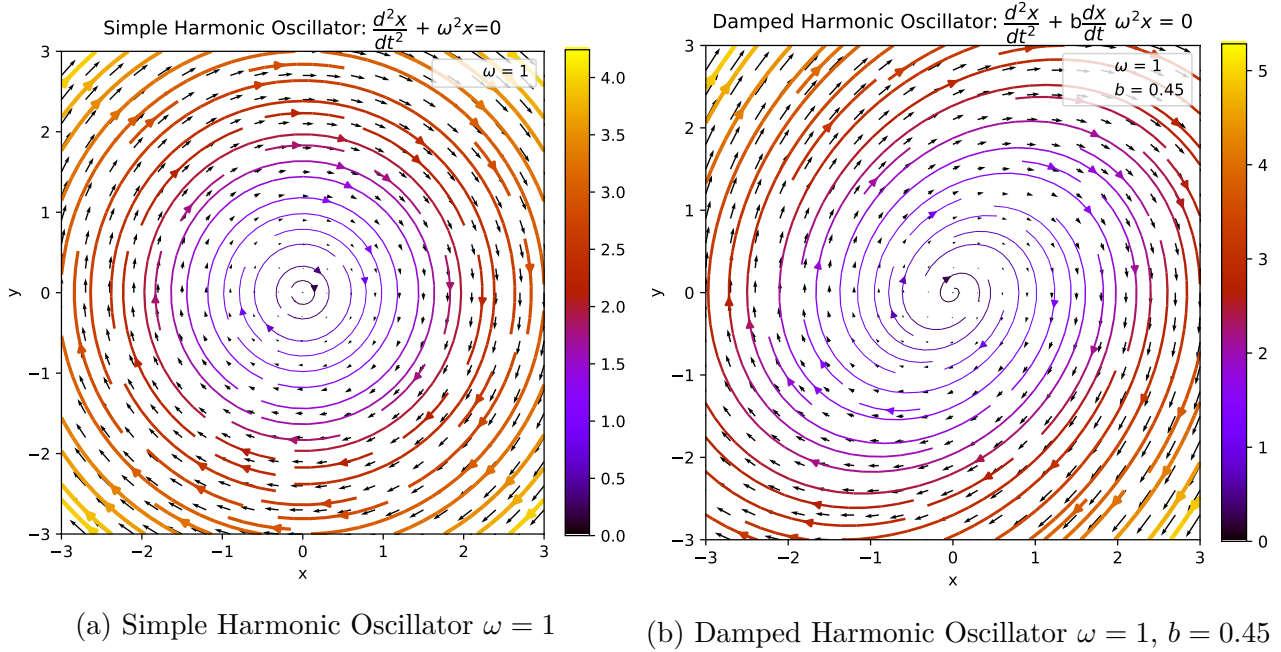


Figure 3: Quiver plot of SHO and Damped SHO

In this example the decision was made to include streamlines as the addition of these extra lines with gradient makes it easier to understand the direction and also the magnitude of the vectors in this example. This plots give us a graphical representation we expect for Simple Harmonic Motion and damped oscillators, when there is no damping involved the vector field is circular with the magnitudes remaining constant (i.e no color change along each individual streamline) meaning that the system is in a constant cycle and the vectors are balanced. When we introduced the damped section we can see that the equation has a decaying nature the vectors begin to point outwards towards the edges as the magnitude of the vectors increases with time meaning the system is no longer being conserved.

Program 4:

In this problem we look at a logistic equation, more specifically the Verhulst model of population. This equation supposedly describes the population of a single species over time. It is supposed to be considered a more complex and accurate depiction of population than a simple exponential population model which was used previously however during this report we will show that even this model is quite simplistic in its approach to determining population. Again the equation is the same as Eq. 1 :

$$\frac{dx}{dt} \equiv \dot{x} = rx\left(1 - \frac{x}{K}\right)$$

We can see that from this expression that the largest value that x can be before the population starts to decay is K which is the carrying capacity until this value is reached we can determine how much the population can still increase by to reach this ratio. As the population increases $1 - \frac{x}{K}$ gets smaller and smaller until it reaches zero and the value x is equal to the value of the carrying capacity. We can reach this value quicker or slower depending on the initial value of r . This is what we will show graphically:

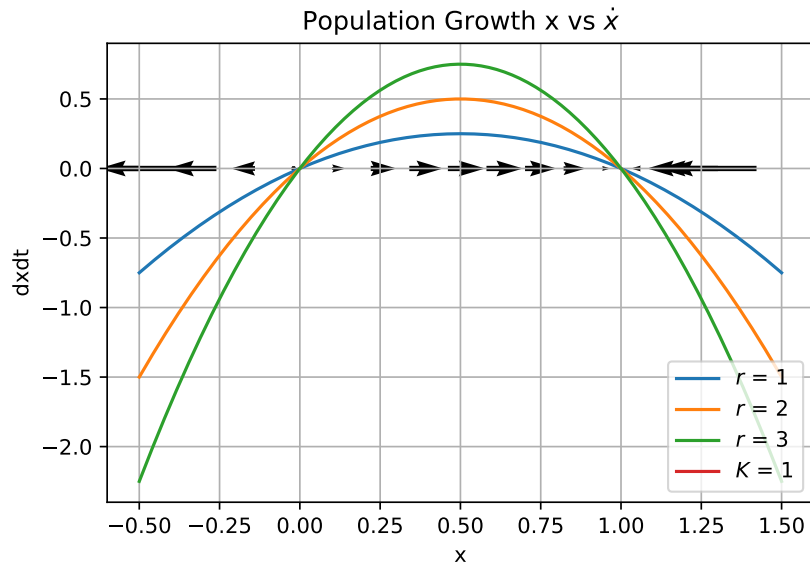


Figure 4: Graphical representation of x vs \dot{x} for Verhulst model

The graph and the one dimensional vector shows all three initial conditions converging at the 0 point on the y axis and the 0 and 1 point on the x axis. The point we are interested in is (1,0), this point is where the three initial condition functions converge at the same value of 0 population growth and 1 which is the same value as the carrying capacity K .

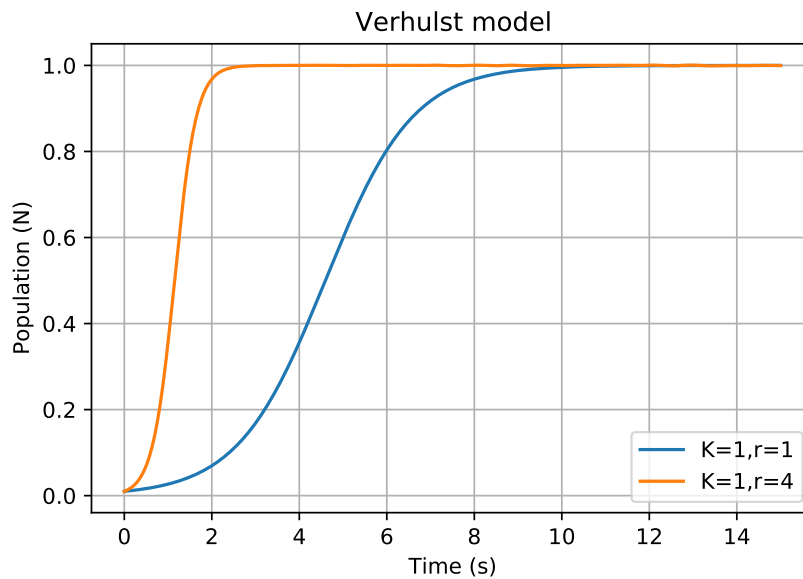


Figure 5: Population Growth over time (Verhulst model)

This graph shows that increasing the r value or the rate of growth increases how fast the population reaches its carry capacity K . This is what we expect to observe. I kept the value of K constant throughout the two graphs however changing this parameter will just change the value the plot will peak at on the y-axis.

Program 5:

In this part of the report we explore our second population equation included in the introduction. This model has already been introduced as the Lotka-Volterra model. As mentioned before this population model differs from the previous one as it explores two populations simultaneously and how they interact with one another. To explore this further we will be using the example of rabbits and foxes.

We start off by making quiver / streamline plots using initial values given:

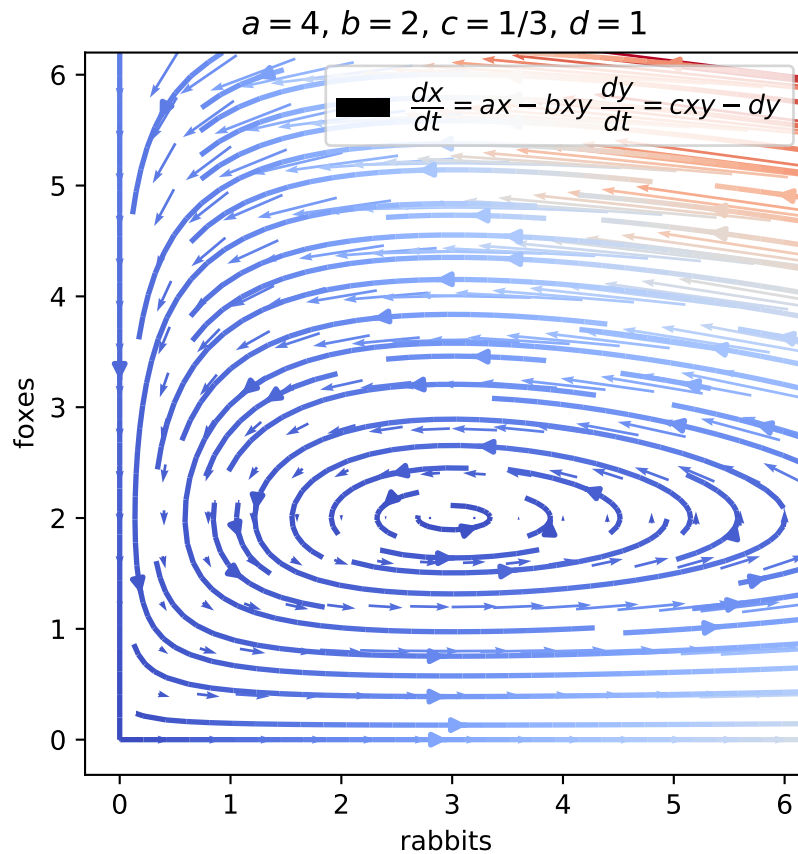


Figure 6: Streamline Quiver plot for Rabbit vs Fox Population

We then repeated this for four more initial conditions a, b, c and d .

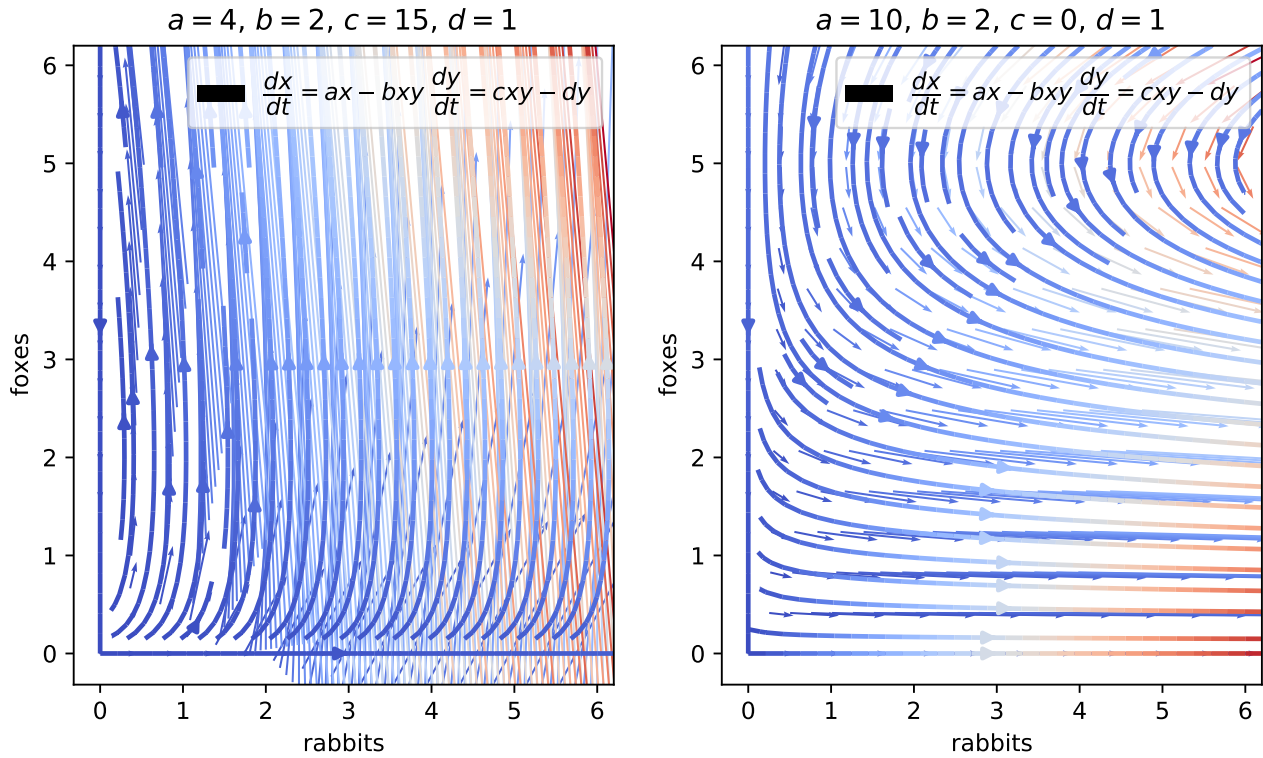


Figure 7: Streamline Quiver plot for Rabbit vs Fox Population

When the c value is disproportionately high, the rabbits are extremely beneficial to the foxes we see long vectors vertically upwards towards the foxes population and its population grows. When the a value is disproportionately high, the growth rate is extremely high the rabbits population increases and the long vectors point horizontally as the rabbits breed in an exponential rate.

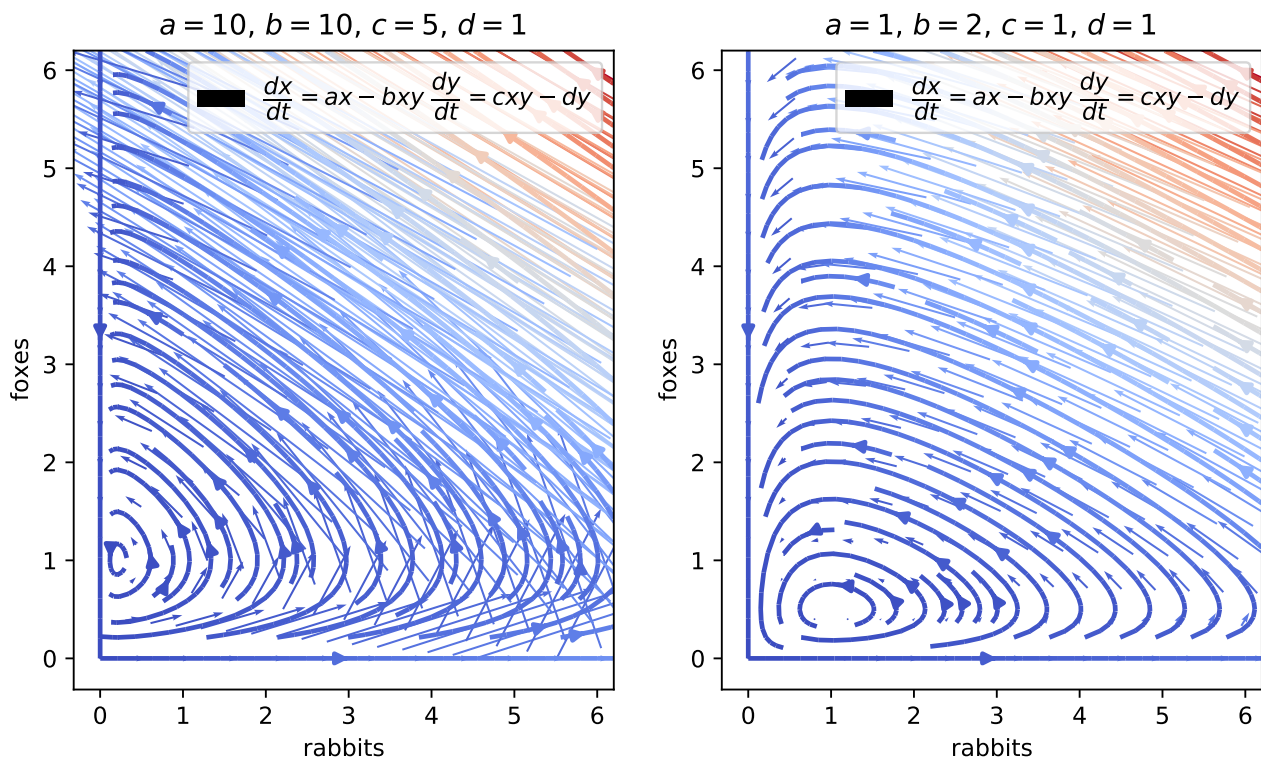


Figure 8: Streamline Quiver plot for Rabbit vs Fox Population

When the growth rate a and b how efficient the foxes are at killing the rabbits are both high the plot skews towards the foxes however the growth rate of the rabbits are good enough to prevent it from being directly vertical vectors.

when everything is equal but b the foxes are more efficient at killing the rabbits the graph skews heavily to the foxes but the population of the foxes does not increase.

We now move on from using vector fields to describe the relationship between the Fox and Rabbit population and the different initial conditions that can be adjusted to affect the outcomes and instead focus our attention on the Runge-kutta RK4 model to solve the system. We can plot these results as ordinary plots and phase diagrams, which will allow us to analyse population size specifically. These graphs are included below:

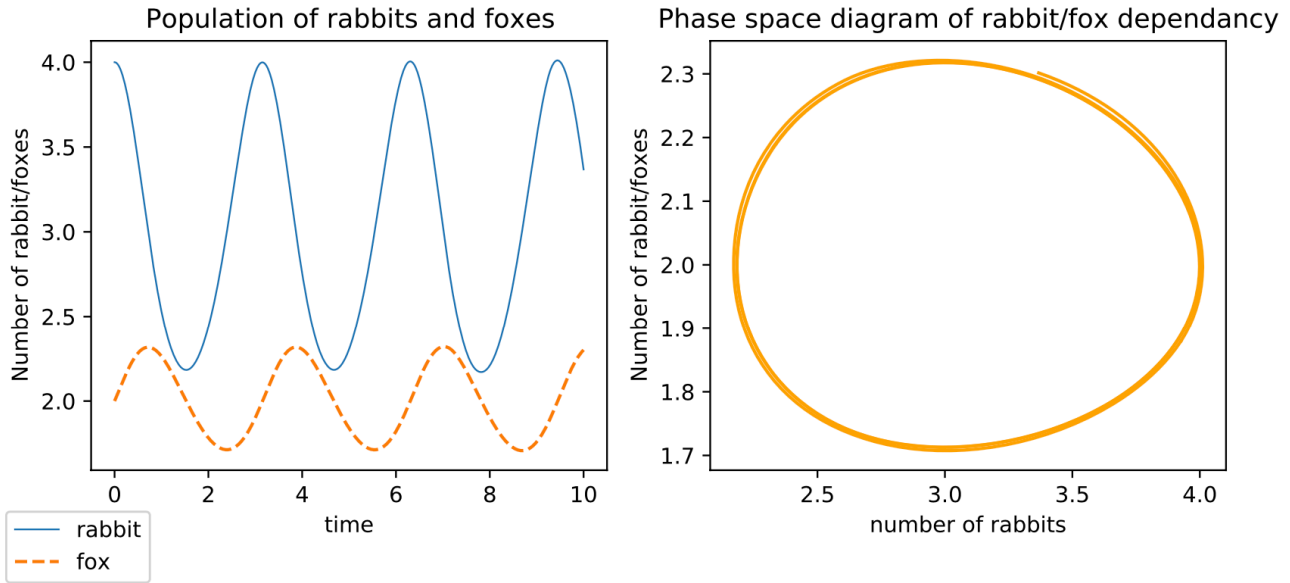


Figure 9: Time series (a) and phase plot of the rabbit and fox population, using $a = 4.0$, $b = 2.0$, $c = 1/3$, and $d = 1.0$ with $x_0 = 4.0$ and $y_0 = 2.0$

This graph describes the relationship that exists between the foxes and rabbits and how they depend on each other as time passes. One thing that is easy to observe is that the fox population is far smaller than the rabbit population and does not increase nearly as quickly as the rabbit population. Looking back at Eq. 2 it's easy to see why, the rabbit population is exactly proportional to the growth rate. Another thing we observe from the graph is that as the fox population rises, the rabbit population drops and vice versa. One thing of note about this graph is that there is no equilibrium where the values come to a steady state, instead it continuously oscillates back and forth between increasing and decreasing each population.

We then repeat this process for the initial values we used above in our quiver plots. (four separate plots)

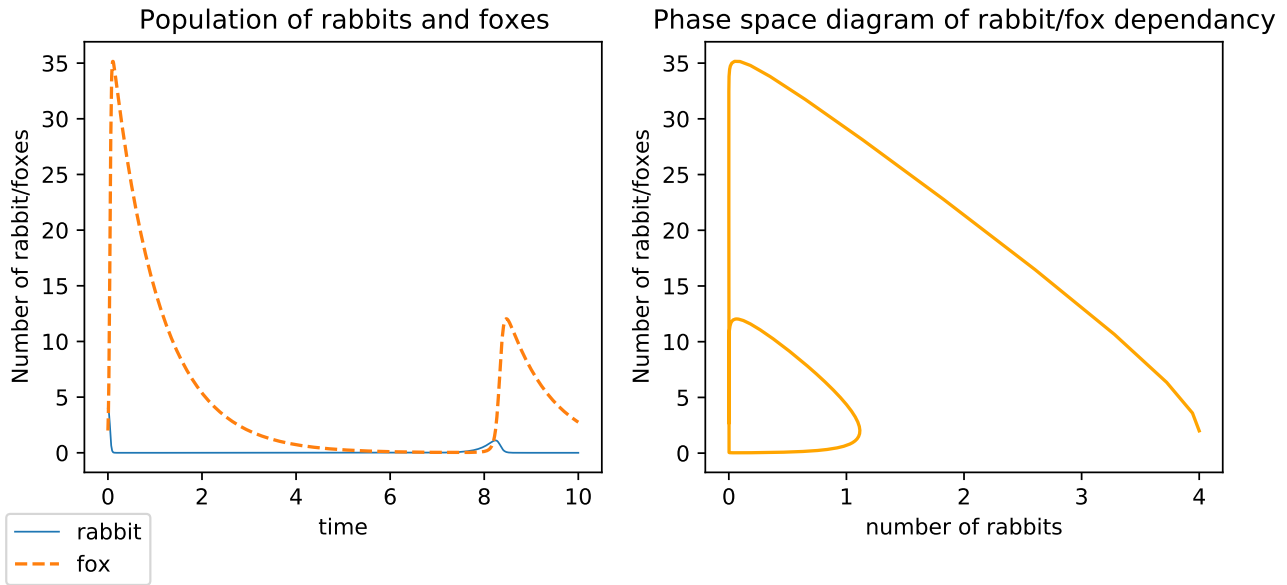


Figure 10: Time series (a) and phase plot of the rabbit and fox population, using $a = 4.0$, $b = 2.0$, $c = 15$, and $d = 1.0$ with $x_0 = 4.0$ and $y_0 = 2.0$

In this graph the rabbits are extremely beneficial to the foxes, at the beginning of the graph the foxes immediately reduce the number of rabbits as they outnumber them so greatly, after some time the fox population dies down as there is no more rabbits, once the fox population reaches zero the rabbits attempt to populate again and are immediately killed by an influx of fox population.

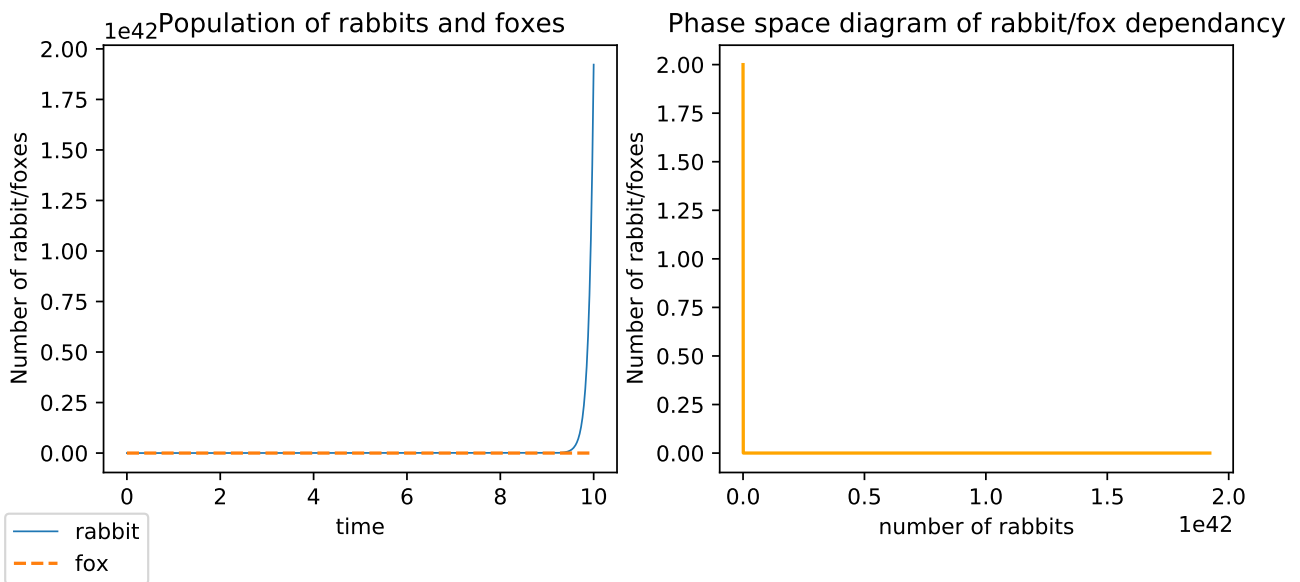


Figure 11: Time series (a) and phase plot of the rabbit and fox population, using $a = 10$, $b = 2.0$, $c = 0$, and $d = 1.0$ with $x_0 = 4.0$ and $y_0 = 2.0$

In this graph the growth population is extremely high and the rabbit population sky rockets exponentially to the point where the graph scales to 1×10^{42} N. The rabbit population also never decreases in the phase diagram.

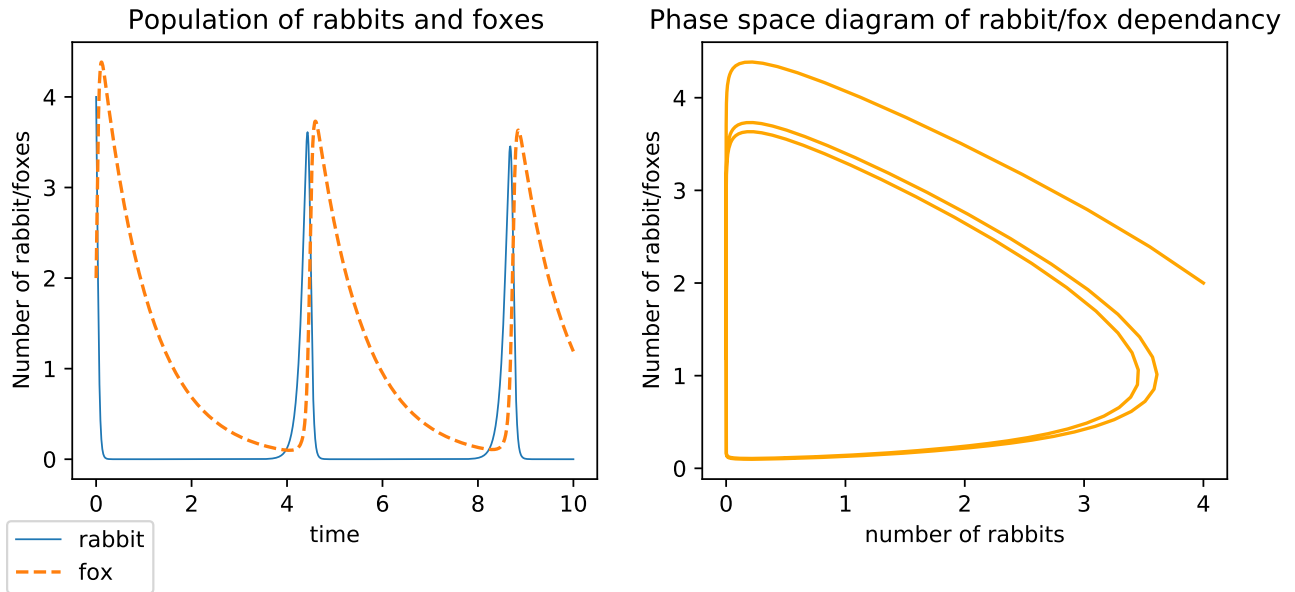


Figure 12: Time series (a) and phase plot of the rabbit and fox population, using $a = 10$, $b = 10$, $c = 5$, and $d = 1.0$ with $x_0 = 4.0$ and $y_0 = 2.0$

In this graph both the growth rate and how efficient the foxes are at killing the rabbits are high, the high populations of the foxes immediately reduces the rabbit population which quickly reduces the foxes population as well, then the rabbit population restores quickly and the fox population follows suit and this continuous on with a reduction in the max population each time.

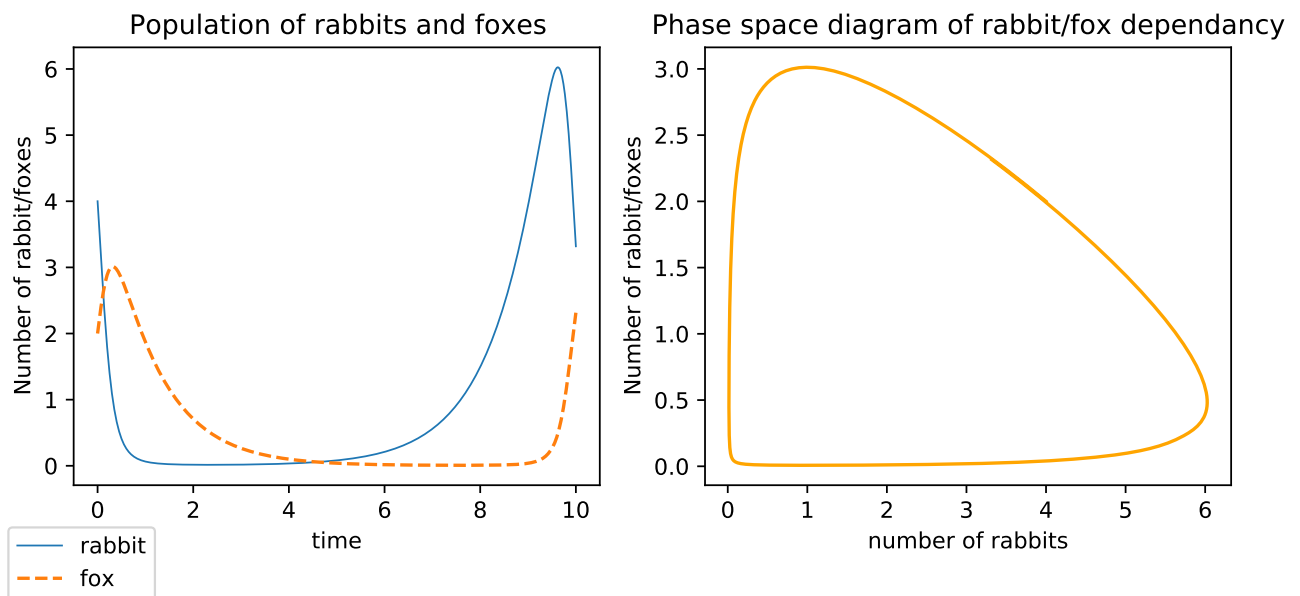


Figure 13: Time series (a) and phase plot of the rabbit and fox population, using $a = 1$, $b = 2$, $c = 1$, and $d = 1.0$ with $x_0 = 4.0$ and $y_0 = 2.0$

In this graph everything is equal but the foxes are more efficient at killing. This shows that the foxes are able to kill the rabbits while in smaller numbers much quicker and will quickly reduce the rabbit population if it increases again.

Program 6:

In the final section of this lab report we will discuss something known as limit cycles. In this example the trajectories will spiral towards or away from this "limit cycle". In order to achieve this affect we need to use the RK45 method once again and we will be using it to solve two connected differential equations. The equations are as follows

$$\frac{dx}{dt} = y$$

$$\frac{dy}{dt} = -x + (1 - x^2)y$$

We will of course be exploring these equations for multiple initial conditions and including graphs of the quiver plot of the graph, the plots of $x(t)$ vs t and $y(t)$ vs t and phase space diagrams of y vs x below for the following initial values:

(a). $x(0) = 3, y(0) = 3;$

(b). $x(0) = -2, y(0) = 2;$

(c). $x(0) = 0.1, y(0) = 0.1;$

(d). $x(0) = 0.5, y(0) = 1;$

As well as 2 other initial parameters:

(e). $x(0) = 0.1, y(0) = 10;$

(f). $x(0) = 5, y(0) = 9;$

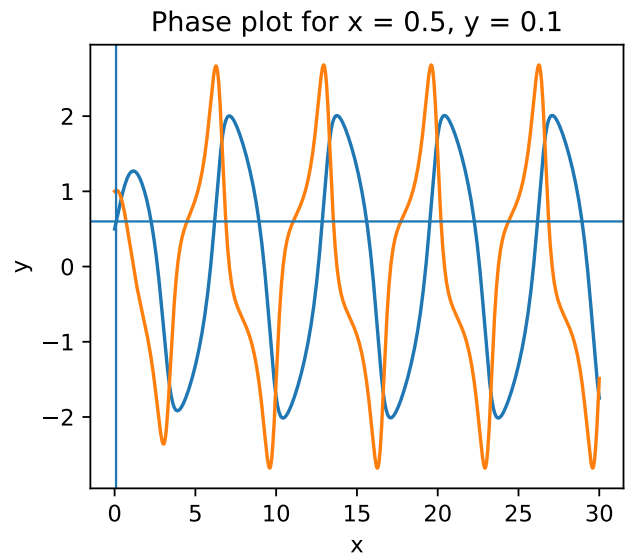
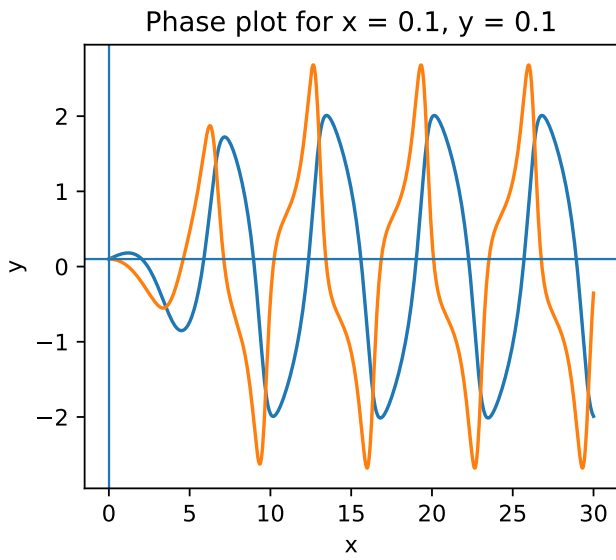
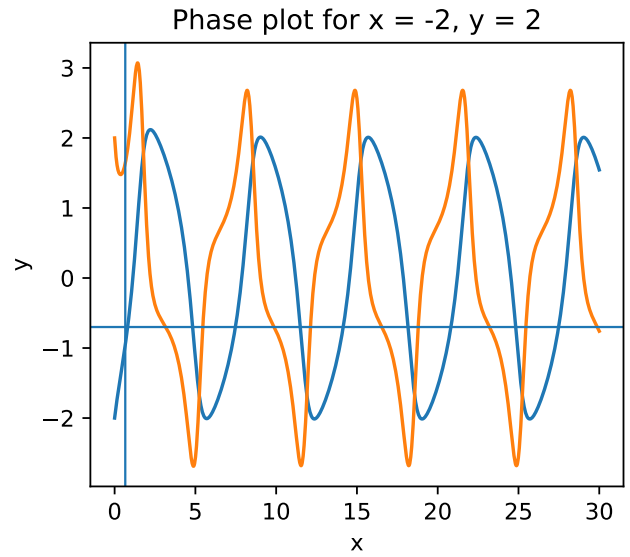
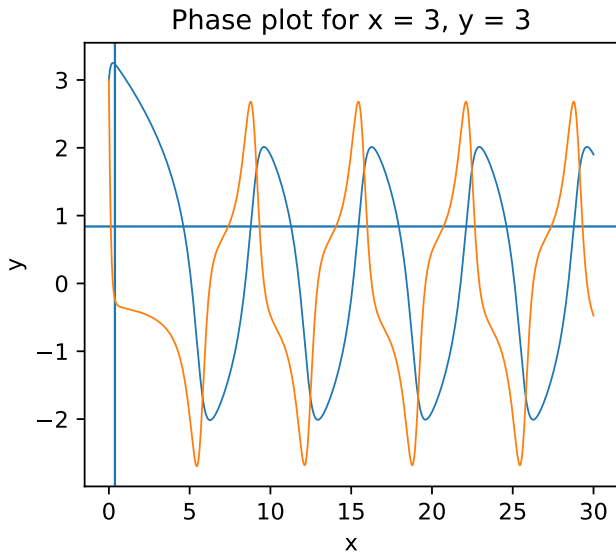


Figure 14: Limit Cycle graphs for different initial values

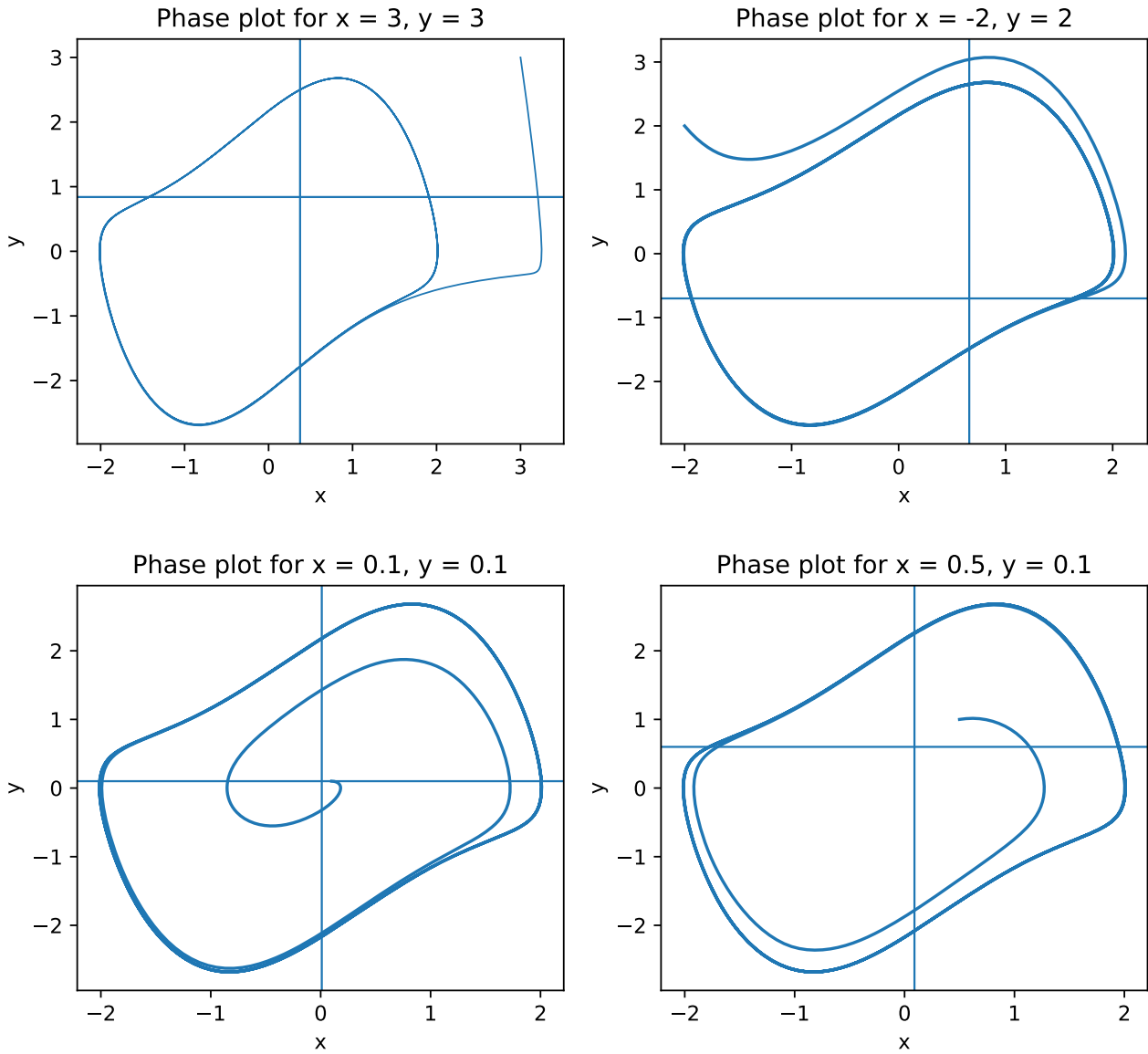


Figure 15: Limit Cycle Phase space diagrams for different initial conditions

From the abundance of graphs included here we can see cyclical behaviour occurring in the plots, in each of the plots the plots start out in an unsteady state and after one cycle they come to a steady state and remain that way for the rest of the cycles. All of the graphs follow the exact same shape and only differ by their initial condition.

From the phase diagrams we can see that the first two phase diagrams are stable limit cycles and spiral into a close loop, the second two phase diagrams are unstable limit cycles as they do not spiral into close loops, therefore the two phase diagrams with the initial conditions we chose are stable limit cycles.

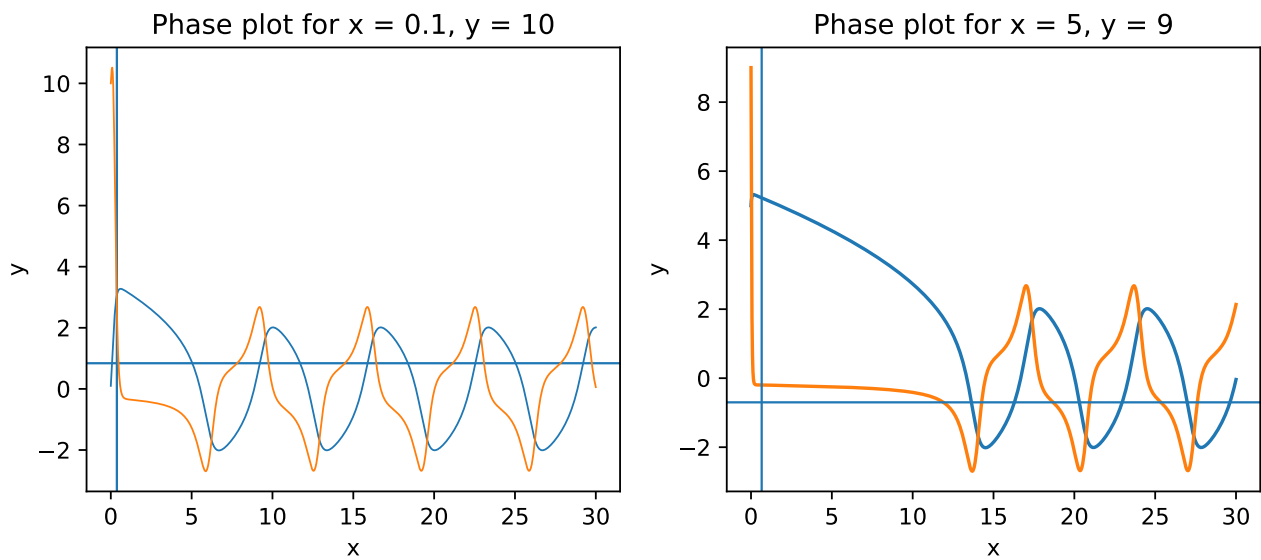


Figure 16: Limit Cycle graphs for different initial conditions

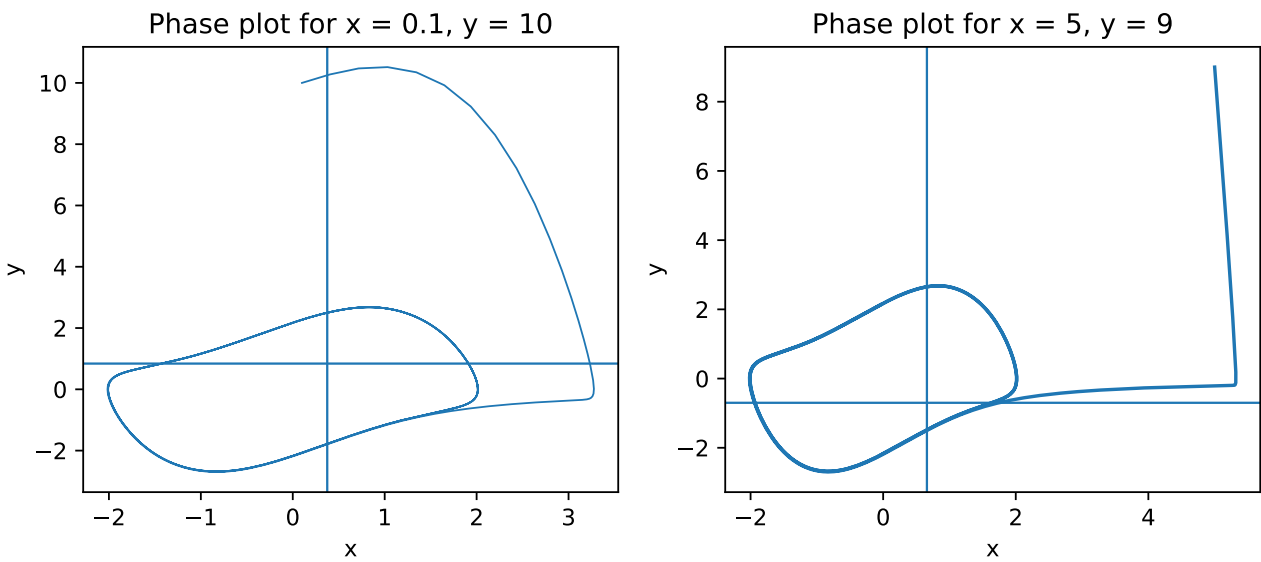


Figure 17: Limit Cycle Phase space diagrams for different initial conditions

4 Discussion and Conclusion

Upon completion of the report we have become familiar with multiple different graphical methods, initially we began with vector fields and using the quiver and streamline methods. These tools allowed us to see the many different solutions possible for differential equations without actually solving these equations. We also used these methods as another way to visualise simple harmonic oscillators and damped harmonic oscillators. We then went on to observe different types of population models, two in particular the Verhulst model (a logistic equation) and the Lotka-Volterra model. We were able to observe both models graphically using quiver and streamline plots and altered the initial conditions to observe the affect each one had on the system.

The Verhulst model's graph allowed us to better understand how the equation describes population growth and observe the assumptions and short coming of the model, particularly if we were to keep expanding our time frame with the model it would no longer accurately represent population, as well as the fact that the model does not account for external factors affecting population.

The Lotka-Volterra model combats the failure of the Verhulst method to account for external factors by primarily focusing on the affect two different species have on the population of each other. We used the example of foxes and rabbits to show that hunters have an inherent affect on the population of its pray and vice versa. This model wasn't perfect as we found that it would always repeat the pattern determined by the plot over time as the population never truly remains at zero. This is one of the big limitations of this model. All of this analysis would not have been possible without the use of quiver and streamline plots to visualise these relationships.

Going one step further with this repeat behaviour concept we explored limit cycles from non-linear differential equations. This showed us that some differential equations mapped to systems which either moved towards being in a stable state and entering a close phase diagram loop or an unstable state where the phase diagram would not be a closed loop. These different outcomes are controlled by whatever initial conditions are chosen for any particular set of non-linear differential equations.